Public-Key Encryption

from the Lattice Isomorphism Problem

Joint work with Adeline Roux-Langlois (CNRS, Greyc, AmacC)
and Alexandre Wallet (Inria, IRISA, Capsule)

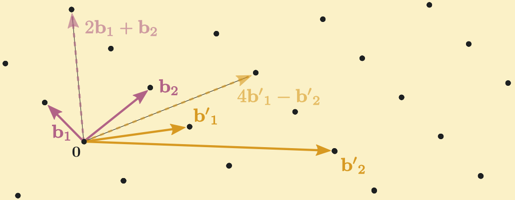
Léo Ackermann

October 2023

Standard lattice-based cryptography

Euclidean lattices

A lattice Λ is a discrete additive subgroup of $\mathbb{R}^n$. It can always be written $\Lambda(\boldsymbol{B}) = \sum_i \boldsymbol{b}_i\mathbb{Z}$.



The more the merrier

The bases are not unique: $\Lambda(\boldsymbol{B}) = \Lambda(\boldsymbol{B})$.

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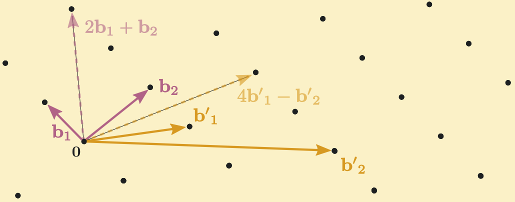

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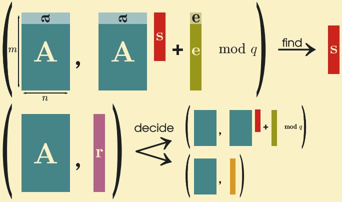
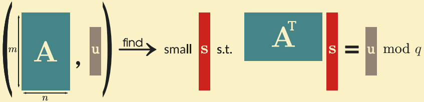
Hard lattice problems

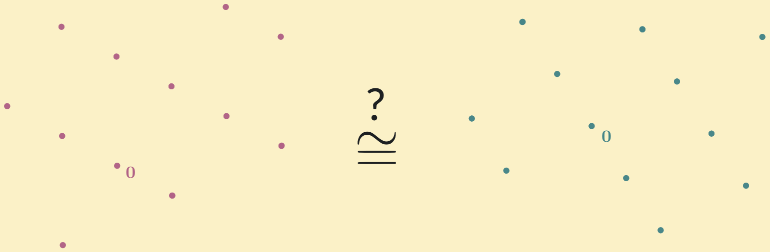
LEARNING WITH ERRORS (LWE).



SHORT INTEGER SOLUTION (SIS).



Lattice Isomorphism Problem (LIP)



Lattice Isomorphism Problem

- Given Λ and Λ', find (if any) $O \in \mathcal{O}(\mathbb{R}^n)$ such that $\Lambda = O \cdot \Lambda'$.

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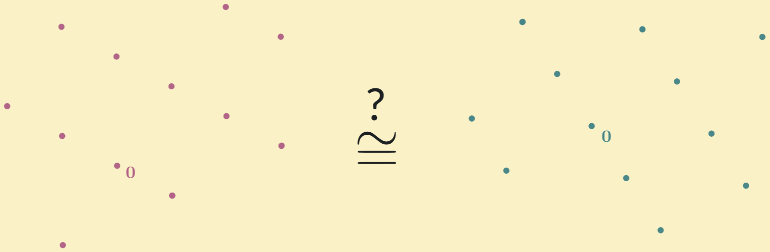

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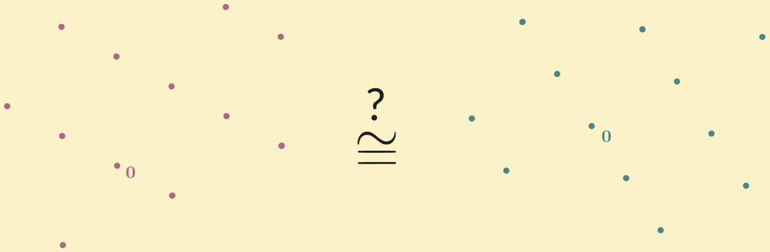

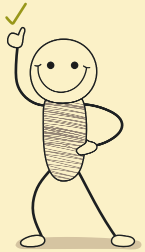

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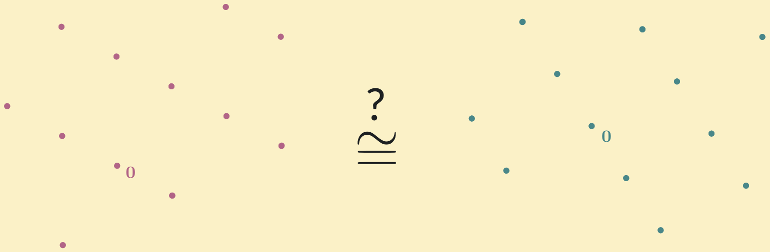

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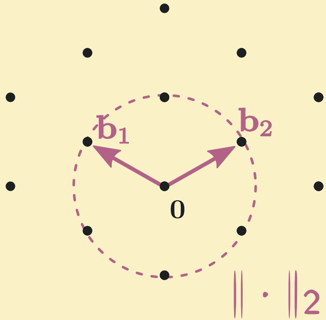
- Given B and B', decide whether $\Lambda(B) \cong \Lambda(B')$ or not. ▷ Decision, dLIP
- Given B, B_0 and B_1, decide whether $\Lambda(B) \cong \Lambda(B_0)$ or $\Lambda(B) \cong \Lambda(B_1)$. ▷ Distinguish, ΔLIP

LIP-based cryptography

LIP flavours

- The *public key* consists in any lattice Λ and a basis B of $O \cdot \Lambda$. The secret key is the rotation O.



LIP-based cryptography

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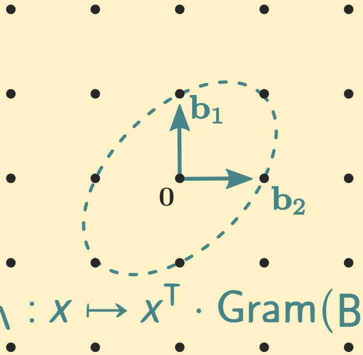
- The *public key* consists in quadratic forms (Q, Q') such that $Q' = U^{\mathsf{T}} Q U$ for $U \in \mathrm{GL}_n(\mathbb{Z})$. The secret key is U.



$$\|\cdot\|_\Lambda : x \mapsto x^{\mathsf{T}} \cdot \mathrm{Gram}(\mathrm{B}) \cdot x$$

where $\mathrm{Gram}(B) = B^{\mathsf{T}} B$

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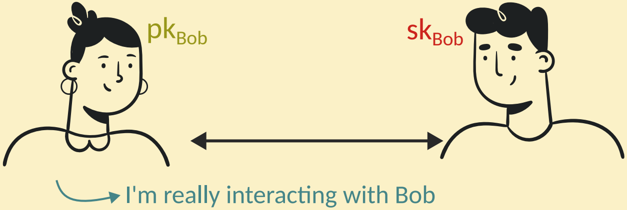
Existing schemes

- Authentication scheme



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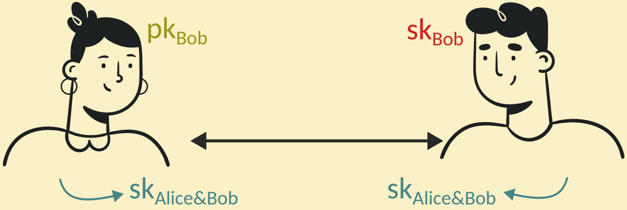

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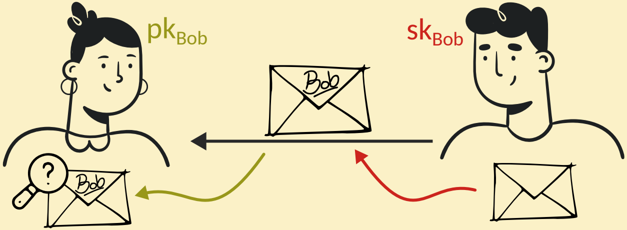

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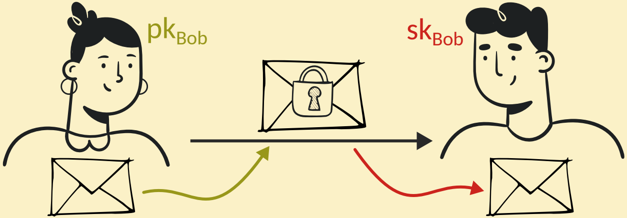
Our work

- Public-key encryption scheme



LIP-based PKE

High-level idea

Follows *Dual-Regev* cryptosystem flavour:

- $\mathcal{C} = (0,1)^n$, $\mathsf{Enc}(0) \sim (D_\Lambda \mod \mathcal{C})$, $\mathsf{Enc}(1) \sim \mathcal{U}(\mathcal{C})$

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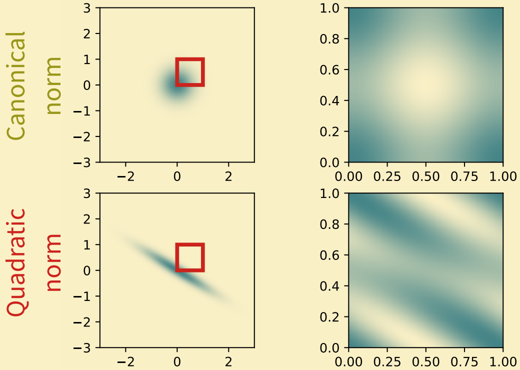

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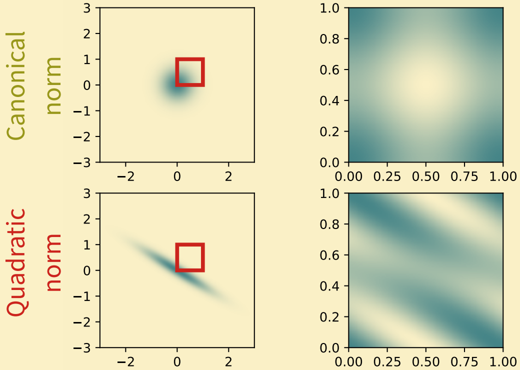

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With overwhelming probability, $\mathsf{Enc}(1)$ is far enough from $(0,1)^n$ vertices.

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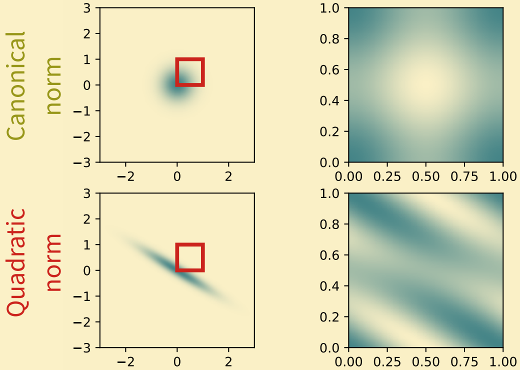

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Security

Under $\Delta LIP_{\mathrm{pke}}$ hypothesis, the scheme is IND-CPA secure.

CAPSULE IRISA AmacC GREYC cnrs

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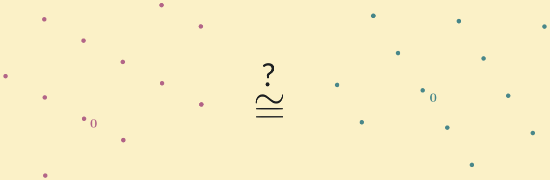

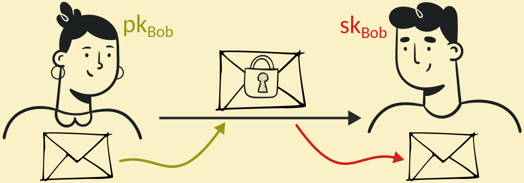

Credits

- Figures are either mine or free pictures from **Freepik**. See e.g. [1], [2].
- The colors are from the **Gruvbox** color palette.
- The E8 lattice comes from [3].