WCC presentation:

Public-key encryption from the lattice isomorphism problem

Presenting **Léo Ackermann** (CNRS, Greyc, Caen)
Joint work with Adeline Roux-Langlois (CNRS, Greyc, Caen)
Alexandre Wallet (Inria, Capsule, Rennes)[1]

[1] Now at PQShield

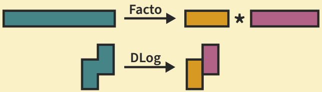
17th June 2024

Lattice-based cryptography

A strong candidate for post-quantum crypto

Cryptographic threat posed by quantum computers

- Shor's algorithm solves the **discrete log** and **factorisation** problems in quantum polynomial time.
- The advent of reasonable quantum computers would **break current cryptosystems** (ECC, RSA).



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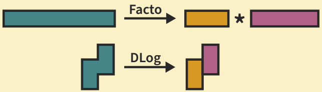


The NIST competition (2016 → 2022)

Three out of four of the first standardized algorithms **rely on lattices**.

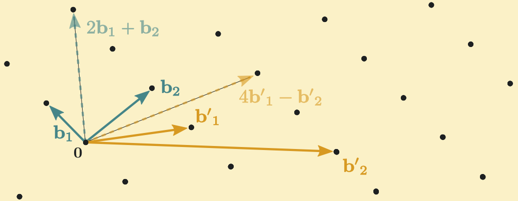
Encryption	Signature
• Crystals-Kyber	• Crystals-Dilithium • Falcon • SPHINCS+

Many lattice-based (and code-based) proposals within the the **extra-round** for **signatures**.

First principles of lattice-based crypto

Euclidean lattices

A lattice Λ is a discrete additive subgroup of $\mathbb{R}^n$.
It can always be written $\Lambda(\boldsymbol{B}) = \sum_i \boldsymbol{b}_i \mathbb{Z}$.



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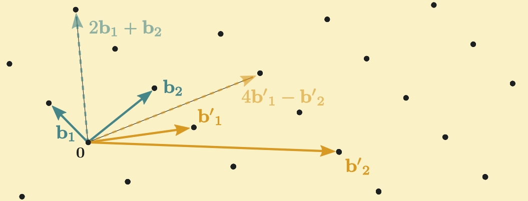

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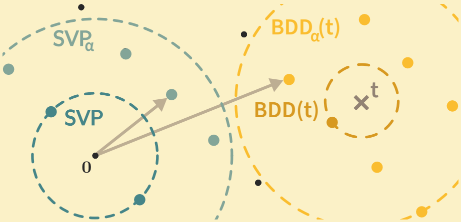
Hard lattice problems



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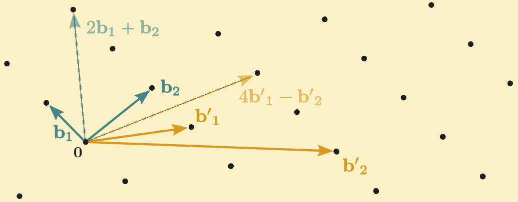

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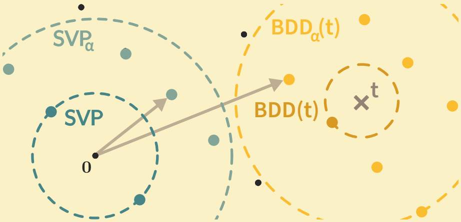


Computational hardness

- $\alpha \in \omega(\sqrt{n}) \Rightarrow \texttt{runtime} \in 2^{\Omega(n)}$.
- $\alpha \in 2^{\Omega(n)} \Rightarrow \texttt{runtime} \in poly(n)$.

Cryptographic assumption

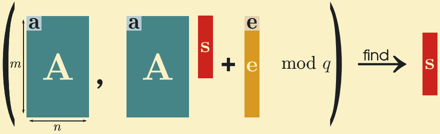
- $\alpha \in poly(n) \Rightarrow \texttt{runtime} \in 2^{\Omega(n)}$.

Lattice-based crypto, legacy approach

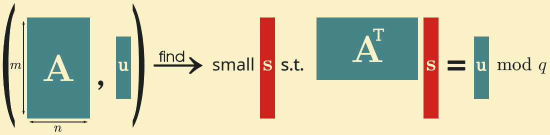
$$\begin{cases} a_{1,1}s_1 & + & \cdots & + & a_{1,n}s_n & = & b_1 \\ \vdots & + & \ddots & + & \vdots & = & \vdots \\ a_{m,1}s_1 & + & \cdots & + & a_{m,n}s_n & = & b_m \end{cases}$$

Lattice-based crypto, legacy approach

Learning With Errors (LWE)



Inhomogeneous Short Integer Solution (ISIS)



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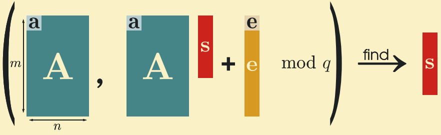

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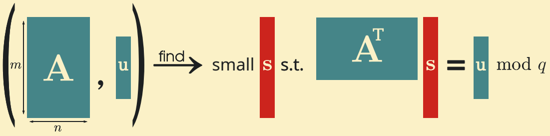

↔ Hardness of LWE and ISIS

Those problems enjoy **worst-case average-case reductions** from hard lattice problems, namely **SVP** and **BDD**.

- LWE $\equiv$ Bounded Distance Decoding over
  $$\Lambda = \{\mathbf{x} \in \mathbb{Z}^m \mid \exists \mathbf{s} \in \mathbb{Z}, \mathbf{x} = \mathbf{A}\mathbf{s} \mod q\}.$$
- ISIS $\equiv$ Shortest Vector Problem over
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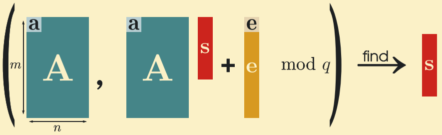

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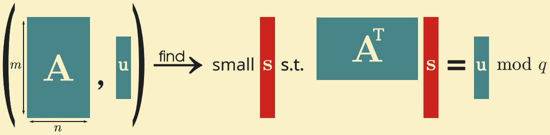

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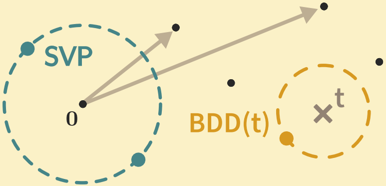
Large variety of constructions

Ranging from simple **encryption** or digital **signature** scheme to **anonymous credentials** and **fully homomorphic encryption**.

Breaking lattice-based crypto

Attacking fundamental lattice problems



SVP Find the *shortest* non-zero vector, of length $\lambda_1(\Lambda) := \min_{\Lambda \setminus \{0\}} \|x\|_2$.

BDD Find v, given a target $t = v + e$, with $v \in \Lambda$ and $\|e\| \leq \lambda_1(\Lambda)/2$.

The **concrete hardness** of those problems is driven by the **gap** $gap(\Lambda)$ between the **actual shortest length** and the upper bound given by **Gaussian heuristics**.

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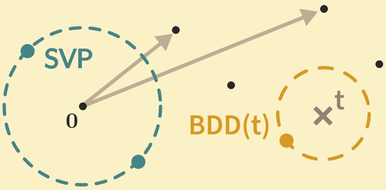

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Deceptive aspect of lattice-based crypto

LWE-lattice: $gap(\Lambda) \geq \Omega(\sqrt{n})$. Prime-lattice: $gap(\Lambda) = \Theta(\log(n))$.

Hypotheses on **random lattices** and subsequent constructions **barely connect** with the luxuriant litterature on **remarkable lattices**.

Public-key encryption from LIP

The lattice isomorphism problem (LIP)

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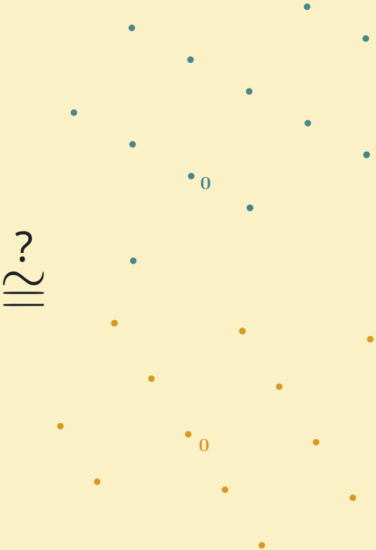

Flavours of lattice isomorphisms

- (Unpractical) Given Λ and Λ', find (if any) $O \in \mathcal{O}(\mathbb{R}^n)$ such that $\Lambda = O \cdot \Lambda'$.

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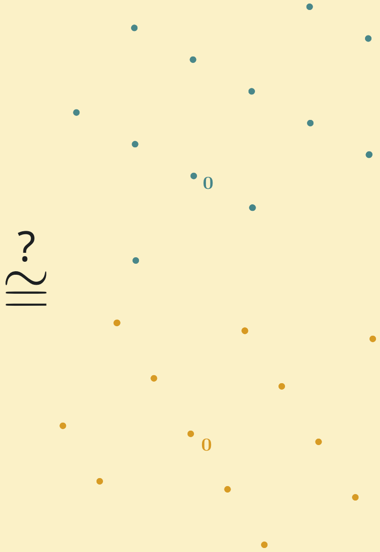

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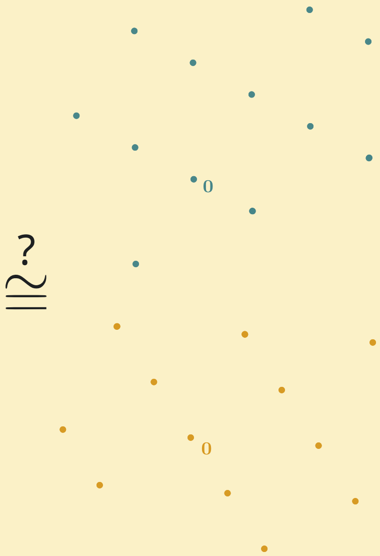

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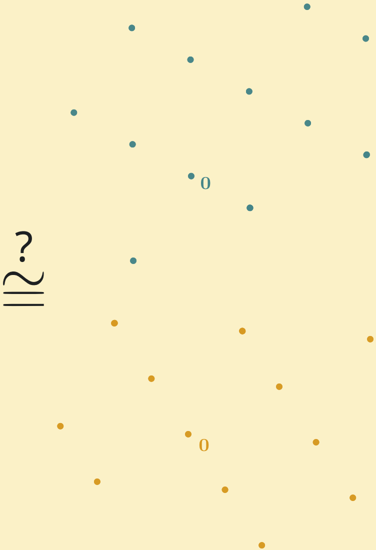

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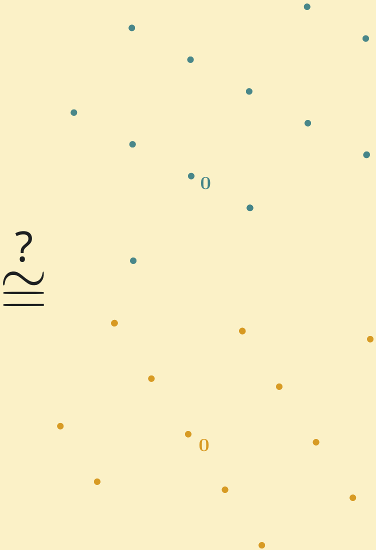

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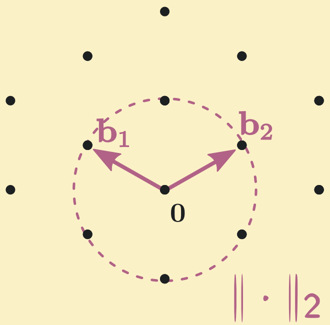
LIP hardness

LIP benefits from **worst-case average-case** self-reduction within an instantiation class, and its **connection with the graph isomorphism problem** accounts for its assumed hardness.

LIP-based crypto

LIP flavours

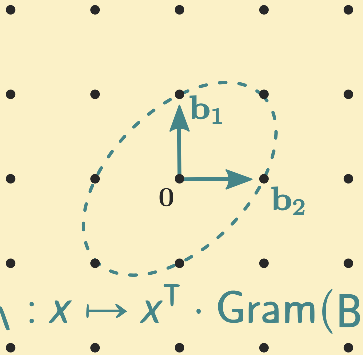
- The **public key** consists in any lattice Λ and a basis $\boldsymbol{B}$ of $O \cdot \Lambda$.
  The **secret key** is the rotation O.



LIP-based crypto

LIP flavours

- The **public key** consists in quadratic forms (Q, Q') such that $Q' = U^{\mathsf{T}} Q U$ for $U \in \mathrm{GL}_n(\mathbb{Z})$. The **secret key** is U.



$$\|\cdot\|_\Lambda : x \mapsto x^{\mathsf{T}} \cdot \mathrm{Gram}(B) \cdot x$$

where $\mathrm{Gram}(\mathbf{B}) = \mathbf{B}^{\mathsf{T}}\mathbf{B}$.

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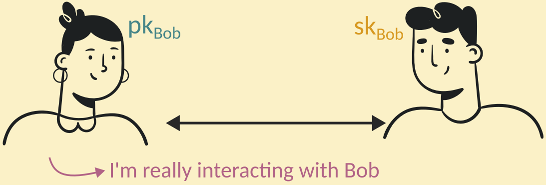
Existing constructions

- Authentication scheme



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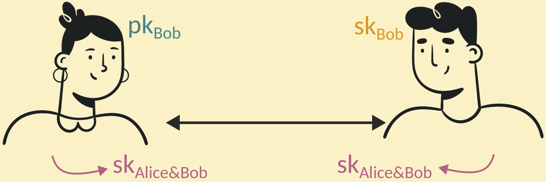

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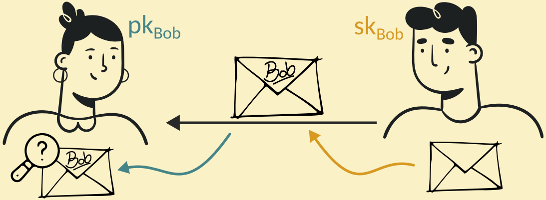
- Signature (including Hawk submission)



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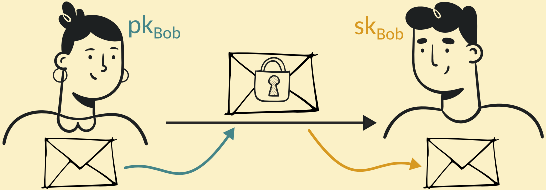

A missing primitive

We propose the first **direct construction** of a PKE relying on **LIP**.

LIP-based public-key encryption scheme

High-level idea

Follows **Dual-Regev** cryptosystem flavour:

- $\mathcal{C} = (0,1)^n$, $\mathsf{Enc}(0) \sim (D_\Lambda \mod \mathcal{C})$, $\mathsf{Enc}(1) \sim \mathcal{U}(\mathcal{C})$.

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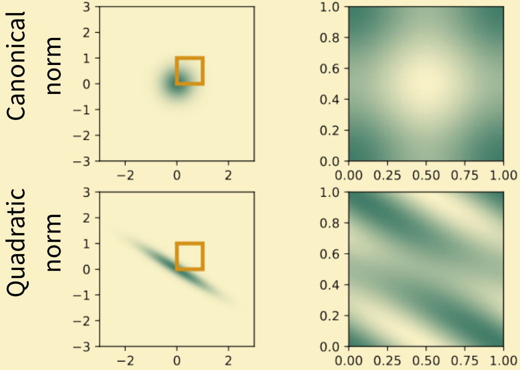

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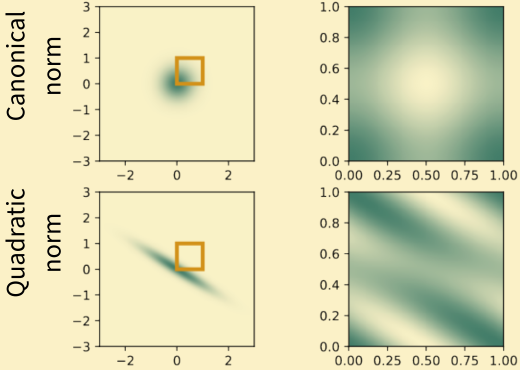

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Correctness

With overwhelming probability, $\mathrm{Enc}(1)$ is far enough from $(0,1)^n$ vertices.

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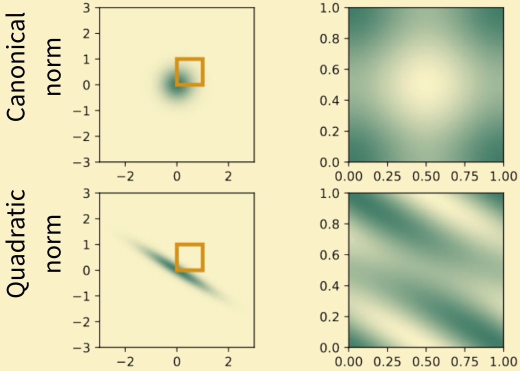

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Security

Under $\Delta LIP_{\mathrm{pke}}$ hypothesis, the scheme is **IND-CPA** secure.

Cryptanalysis of the $\Delta\mathrm{LIP}_{\mathrm{pke}}$ hypothesis

A reasonable hypothesis

The $\Delta\mathrm{LIP}_{\mathrm{pke}}$ hypothesis seems as strong as $\Delta\mathrm{LIP}$: the class restriction **does not improve existing attacks**, and **does not create new ones** neither.

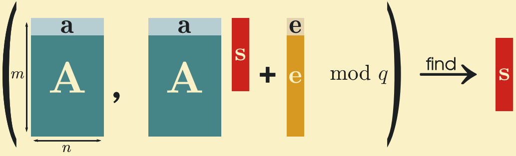
A reasonable conjecture for falsifiability

For $n \geq 85$, there exists at least one unimodular lattice Λ of rank n that verifies $\lambda_1(\Lambda)^2 \geq \sqrt{72n}$.

Discussion

What could we expect from LIP?

Legacy approach



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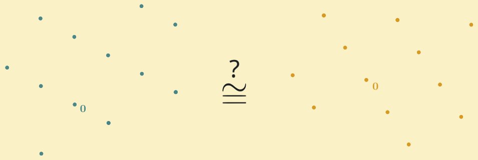

Large *gap*(Λ) on random lattices
Hard time with Gaussian sampling
Plenty of constructions
Approximate variants of hypotheses

Small *gap*(Λ) on remarkable lattices
Easy implementation (eg. Hawk)
Only a few constructions
Fragile hypotheses

Thank you for your attention!

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